

# PROBABILISTIC SPECTRAL SPARSIFICATION IN SUBLINEAR TIME

Yin Tat Lee  
yintat@mit.edu  
MIT

**ABSTRACT.** In this paper, we introduce a variant of spectral sparsification, called probabilistic  $(\varepsilon, \delta)$ -spectral sparsification. Roughly speaking, it preserves the cut value of any cut  $(S, S^c)$  with an  $1 \pm \varepsilon$  multiplicative error and a  $\delta |S|$  additive error. We show how to produce a probabilistic  $(\varepsilon, \delta)$ -spectral sparsifier with  $O(n \log n / \varepsilon^2)$  edges in time  $\tilde{O}(n / \varepsilon^2 \delta)$  time for unweighted undirected graph. This gives fastest known sub-linear time algorithms for different cut problems on unweighted undirected graph such as

- An  $\tilde{O}(n / OPT + n^{3/2+t})$  time  $O(\sqrt{\log n / t})$ -approximation algorithm for the sparsest cut problem and the balanced separator problem.
- A  $n^{1+o(1)} / \varepsilon^4$  time approximation minimum s-t cut algorithm with an  $\varepsilon n$  additive error.

## 1. INTRODUCTION

Many cut-based graph problems can be solved approximately in time  $m^{1+o(1)}$ , such as the sparsest cut problem, the balanced separator problem, the minimum s-t cut problem. For dense graphs, we can approximate graphs by sparse graphs and obtain  $O(m) + n^{1+o(1)}$  time approximation algorithms for different cut-based problems. Unfortunately, in the era of big data, many dense graphs are too large to process explicitly, such as distance matrices in machine learning. It is natural to ask whether it is possible to approximately solve cut-based graph problems on these graphs in sublinear time.

**1.1. Previous results on sublinear time algorithm for optimization problems.** There are many results on estimating the optimum value of various combinatorial problems in sublinear time, such as maximum matching [24, 31], minimum vertex cover [23, 26, 31] and minimum set cover [24, 31]. Many of these algorithms simulate [24] classical approximation algorithms using local information and transform the classical algorithms into constant-time algorithms. The running time of these constant-time algorithms usually depends exponentially on the maximum degree of the graph and the additive error  $\delta$ . Unfortunately, there has been little progress for dense graphs because of the limitation of this simulation approach. The only result for dense graphs we aware of is an  $\tilde{O}(n \cdot \text{poly}(1/\varepsilon))$  time algorithm for finding an factor-2 approximation of the size of a maximum vertex cover within an extra  $\varepsilon n$  additive error [25].

Instead of using the simulation approach, we suggest another principled way to obtain sublinear time algorithms - sparsification.

**1.2. Sparsification.** In this work, we heavily use the concept of sparsification from the spectral graph theory. Benczúr and Karger [2] introduced the notation of cut sparsification for solving cut-based problem on dense graphs, but it is not designed for sublinear time algorithms. A graph  $H$  is called a cut sparsifier of  $G = (V, E, \omega)$  if  $H$  is a sparse graph on  $V$  such that the cut value of any cut in  $H$  is within a factor of  $(1 \pm \varepsilon)$  of its value in  $G$ . In other

words, for all characteristic vectors  $x \in \{0, 1\}^V$ , we have

$$(1.1) \quad \sum_{u \sim v} \tilde{\omega}_{uv} (x(u) - x(v))^2 \in (1 \pm \varepsilon) \sum_{u \sim v} \omega_{uv} (x(u) - x(v))^2$$

where  $\omega$  and  $\tilde{\omega}$  are the weights of edges in graph  $G$  and  $H$  respectively. They proved that sampling a graph with certain probability gives a cut sparsifier and the sampling probability can be computed in time  $\tilde{O}(m)$ . This gives an  $\tilde{O}(m)$  time algorithm to find a cut sparsifier with  $\tilde{O}(n/\varepsilon^2)$  edges. [9, 10, 11] used the cut sparsification to obtain various fast algorithms for the minimum s-t cut problem and the maximum flow problem for dense graphs. Besides this, cut sparsification has many other applications because of its strong guarantee. Of particular relevance to this paper, Mařdry [22] used the cut sparsification as one of the essential components to give a way to reduce cut problems on general graph to some almost trees and obtained almost linear time algorithms for many cut problems.

Inspired by the cut sparsification, Spielman and Teng [30] defined the notation of spectral sparsification, which is a stronger notation of sparsification. It requires the graph  $H$  satisfies (1.1) for all vectors  $x \in \mathbb{R}^V$ . From numerical perspective, it is same as requiring the Laplacian of the graph  $H$  is a good preconditioner of the Laplacian of the graph  $G$ . So, many equations related to the Laplacian  $G$ , such as, Laplacian equation, eigenvalue problem, heat equation, random walk, can be solved in the graph  $H$  within a certain error. Spielman and Srivastava [29] showed that spectral sparsifiers can be found by sampling the graph with probability proportional to effective resistances. And they presented an algorithm to estimate effective resistances in time  $\tilde{O}(m)$  using nearly linear time Laplacian solver [16, 14, 18].

Although there are a lot of results for the streaming model [7, 12, 8], there is no sublinear time algorithm because it is apparently impossible.

**1.3. Our contribution.** Motivated by the sublinear time problem and the spectral graph theory, we introduce a variant of spectral sparsification [30] that we call probabilistic spectral sparsification. Given an unweighted graph  $G = (V, E)$ , a probabilistic  $(\varepsilon, \delta)$ -spectral sparsifier of the graph  $G$  is a weighted random graph  $\tilde{G} = (V, \tilde{E}, \tilde{\omega})$  on the vertex set  $V$  such that

(1) Lower Bound: We have

$$(1.2) \quad (1 - \varepsilon) \sum_{(x,y) \in E} (u(x) - u(y))^2 \leq \sum_{(x,y) \in \tilde{E}} \tilde{\omega}(x,y) (u(x) - u(y))^2 \quad \text{for all } u \in \mathbb{R}^V.$$

(2) Upper Bound <sup>1</sup>: For all  $u \in \mathbb{R}^V$ , we have

$$(1.3) \quad \sum_{(x,y) \in \tilde{E}} \tilde{\omega}(x,y) (u(x) - u(y))^2 \leq (1 + \varepsilon) \sum_{(x,y) \in E} (u(x) - u(y))^2 + \delta \|u\|_2^2 \quad \text{with high probability.}$$

It seems to us that standard matrix concentration bound can at best give bounds like  $\delta \sum d(x) u^2(x)$  and there are results [5, 6] on this line concerning fast approximate general matrix without paying  $\tilde{O}(m)$  time to compute effective resistances. However, the guarantee  $\delta \sum d(x) u^2(x)$  can be  $n$  times worse than  $\delta \|u\|_2^2$  for dense matrices and it is not good enough for certain applications such as the sparsest cut problem.

In this paper, we show how to construct a probabilistic  $(\varepsilon, \delta)$ -spectral sparsifier with  $\tilde{O}(n/\varepsilon^2)$  edges in time  $\tilde{O}(n/\varepsilon^2 \delta)$ . We avoid the matrix concentration bound by using graph structures and obtain this almost tight result. As a result, this transforms many cut problems on dense graphs into sparse graphs and hence gives sublinear algorithms on a bunch of cut-based

<sup>1</sup>In this paper, high probability means a constant probability sufficiently close to 1.

problems. We illustrate the applicability of our sparsification on the following fundamental cut-based graph problems

- An  $\tilde{O}(n/OPT + n^{3/2+t})$  time  $O(\sqrt{\log n/t})$ -approximation algorithm for the sparsest cut problem and the balanced separator problem.
- An  $\tilde{O}\left(n/OPT + 2^k n^{1+1/(3 \cdot 2^k - 1) + o(1)}\right)$  time  $O\left(\log^{(1+o(1))(k+1/2)} n\right)$ -approximation algorithm for the sparsest cut problem and the balanced separator problem.
- An  $\tilde{O}(\sqrt{mn}/\varepsilon^3)$  time and a  $n^{1+o(1)}/\varepsilon^4$  time approximation minimum s-t cut algorithm with an  $\varepsilon n$  additive error.

This sparsifier is a weaker notion than the spectral sparsification introduced by Spielman and Teng [30], which requires a single graph to satisfy both upper and lower bounds with  $\delta = 0$ . To justify our notion, we show that it takes at least  $\Omega(n/\varepsilon^2 + n/\delta)$  time to find this sparsifier and hence the extra additive term is unavoidable. Furthermore, we show in Theorem 12 that the term  $n/OPT$  in the running time shown above is unavoidable for the sparsest cut problem.

**1.4. Definitions.** Let  $[n] = \{1, 2, \dots, n\}$ . The notation  $\tilde{O}(f(n))$  means  $O(f(n) \log^c(n))$  for some constant  $c$  and  $\tilde{\tilde{O}}(f(n))$  means  $O(f(n) \log^c \log(n))$  for some constant  $c$ . Let  $G$  be a weighted undirected graph with  $n$  vertices and  $m$  edges with weights  $\omega$ . We write  $(u, v) \in G$  if the vertex  $u$  is adjacent to the vertex  $v$  in the graph  $G$ . Let the neighborhood of  $v$  be  $N_G(v) \stackrel{\text{def}}{=} \{u : (u, v) \in G\}$ . Let  $d_G(u)$  be the weighted degree of the vertex  $u$ , that is  $d_G(u) = \sum_{(u,v) \in G} \omega(u, v)$ . The cut value of  $U$  is defined by  $\text{Cut}_G(U) = \sum_{(u,v) \in G, u \in U, v \notin U} \omega(e)$ .

**Definition 1.** Given a weighted undirected graph  $G$ , we view the graph  $G$  as an electric network and define the resistance of an edge  $(s, t)$  is  $1/\omega(s, t)$ . The effective resistance  $R(s, t)$  is the potential difference between  $s$  and  $t$  when there is a unit flow send from  $s$  to  $t$  on this electric network.

**Definition 2.** (General Graph Model) In the general graph model, a graph  $G = (V, E)$  is represented by the number of vertices  $n$  and three oracles

- (1) The vertex oracle  $\mathcal{O}_1 : [n] \rightarrow V$  which returns the  $i$ -th vertex of the graph.
- (2) The degree oracle  $\mathcal{O}_2 : V \rightarrow \mathbb{Z}^+$  which returns the degree  $d(v)$ .
- (3) The edge oracle  $\mathcal{O}_3 : V \times \mathbb{Z}^+ \rightarrow V$  which returns the  $i$ -th vertex adjacent to  $v$ .

## 2. PROBABILISTIC SPECTRAL SPARSIFICATION

In this section, we show how to construct probabilistic spectral sparsifiers in sublinear time. The algorithm is inspired by the following two results about effective resistance. Spielman and Srivastava [29] shows that sampling edges proportional to the effective resistances of edges produce a spectral sparsifier. It is known that on an unweighted expander, we have [20]

$$(2.1) \quad R(s, t) = \Theta\left(\frac{1}{d(s)} + \frac{1}{d(t)}\right)$$

for any edge  $(s, t)$ . These two results show that we can construct spectral sparsifiers for expanders according to the degree of vertices. Therefore, if we can transform a graph into an expander by modifying only some edges, then we can obtain a spectral sparsifier with small additive error. Unfortunately, it requires modifying  $O(m)$  edges which is too large for certain problems. Instead of satisfying the expander condition for (2.1), we show how to make a graph satisfies (2.1) directly by adding only a few edges. To do this, we randomly select a subset of the graph and put a sparse expander on this subset. In Lemma 4, we show that the effective

resistances in this new graph satisfies the estimate (2.1). This gives an algorithm to construct probabilistic spectral sparsifiers.

In this paper, the only property of expander we used is that there are lots of edge-disjoint short paths in an expander.

**Theorem 3.** [21, 6] *There is an  $O(n)$  time algorithm to construct a graph  $E_n$  such that*

- (1) *It has  $\Theta(n)$  vertices,  $O(n)$  edges and the maximum degree is  $\Theta(1)$ .*
- (2) *For any pairs  $\{(a_i, b_i)\}_{i=1}^k$  with  $k = O(\frac{n}{\log n})$ , there exists edge-disjoint paths of length  $O(\log n)$  in  $E_n$  joining  $a_i$  to  $b_i$ .*

The following key lemma shows that putting  $E_n$  in a random subset of  $G$  makes the graph satisfies (2.1).

**Lemma 4.** *Assume  $\delta \leq 1/\log n$ . Given an unweighted undirected graph  $G = (V, E)$ . Let  $E_{\delta n}$  be the graph given by Theorem 3 and  $V_\delta$  be a random subset of  $V$  with size  $|E_{\delta n}|$ . We view  $E_{\delta n}$  as a graph on  $V_\delta$  and let  $\tilde{G}$  be the union of  $G$  and  $E_{\delta n}$ . With high probability, for any edge  $(s, t)$ , we have*

$$\frac{1}{2} \left( \frac{1}{d_{\tilde{G}}(s)} + \frac{1}{d_{\tilde{G}}(t)} \right) \leq R_{\tilde{G}}(s, t) \leq O \left( \frac{\log n}{\delta} \left( \frac{1}{d_{\tilde{G}}(s)} + \frac{1}{d_{\tilde{G}}(t)} \right) \right).$$

*Proof.* Claim: With high probability, for any vertex  $v$  with  $d_{\tilde{G}}(v) = \Omega(\log n/\delta)$ ,

$$|V_\delta \cap N_G(v)| = \Omega(\delta d_{\tilde{G}}(v)).$$

Assume the claim. Let  $(s, t)$  be any edge. Write  $d_{\tilde{G}}(s)$  as  $d(s)$  and  $d_{\tilde{G}}(t)$  as  $d(t)$  for simplicity. Since the effective resistance of an edge is bounded by 1 for unweighted graph, if  $d(s)$  or  $d(t)$  is at most  $O(\log n/\delta)$ , we have

$$R_{\tilde{G}}(s, t) \leq 1 = O \left( \frac{\log n}{\delta} \left( \frac{1}{d(s)} + \frac{1}{d(t)} \right) \right).$$

Hence, we can assume both  $d(s)$  and  $d(t)$  is at least  $\Omega(\log n/\delta)$ . The claim shows that there are at least  $\Omega(\delta d(s))$  vertices of  $V_\delta$  in the neighbor  $N_G(s)$  of  $s$  and at least  $\Omega(\delta d(t))$  for  $t$ . Since  $\delta d(s) \leq n/\log n$ , Theorem 3 shows that there are  $\Omega(\delta \min(d(s), d(t)))$  edge-disjoint paths with length  $O(\log n)$  joining these neighbor of  $s$  to these neighbor of  $t$ . By Rayleigh's Monotonicity Principle, the effective resistance between  $s$  and  $t$  is less than the graph with only  $\Omega(\delta \min(d(s), d(t)))$  edge-disjoint paths from  $s$  to  $t$  with length  $O(\log n)$ . Hence, we have

$$R_{\tilde{G}}(s, t) = O \left( \frac{\log n}{\delta \min(d(s), d(t))} \right) = O \left( \frac{\log n}{\delta} \left( \frac{1}{d(s)} + \frac{1}{d(t)} \right) \right).$$

Therefore, in both case, we have

$$R_{\tilde{G}}(s, t) \leq O \left( \frac{\log n}{\delta} \left( \frac{1}{d(s)} + \frac{1}{d(t)} \right) \right).$$

Another side of the inequality comes from [20].

Proof of the claim: Let  $U$  be any subset of  $V$  with  $k$  elements. Note that  $X = |V_\delta \cap U|$  is a random variable with hypergeometric distribution. The Chernoff bound for hypergeometric distribution [4, Thm 1.17] shows that  $\mathbb{P}(X \leq \frac{1}{2}\mathbb{E}(X)) \leq \left(\frac{2}{e}\right)^{\mathbb{E}(X)/2}$ . For  $k = \Omega(\log n/\delta)$ , we have  $\mathbb{E}(X) = \delta k = \Omega(\log n)$  and hence  $\mathbb{P}(X \leq \frac{\delta k}{2}) = \frac{1}{\text{poly}(n)}$ . Since there are only  $n$  neighbor sets  $N_G(v)$ , union bound shows that with high probability, for any  $v \in V$  with  $d_G(v) = \Omega(\log n/\delta)$ , we have

$$|V_\delta \cap N_G(v)| = \Omega(\delta d_G(v)) = \Omega(\delta d_{\tilde{G}}(v))$$

where the last line comes from the fact that the maximum degree of  $E_{\delta n}$  is  $O(1)$ .  $\square$

Having a good estimate of effective resistances, we could use the following algorithm proposed by Spielman and Srivastava [29] to construct a spectral sparsifier of  $\tilde{G}$ .

$H = \mathbf{Sparsify}(G, p, q)$
1. Repeat $q$ times:
1a. Sample an edge $e$ from $G$ with probability $p(e)$ .
1b. Add it to $H$ with weight $(qp(e))^{-1}$ .

**Theorem 5.** [29] *Let  $G$  be an unweighted undirected graph. Suppose  $p(e)$  are numbers such that  $\sum p(e) = 1$  and*

$$p(e) \geq \frac{R(e)}{\alpha n}$$

*for some  $\alpha > 0$ . Then, with high probability,  $\mathbf{Sparsify}(G, p, \Theta(\alpha n \log n / \varepsilon^2))$  is a  $\varepsilon$ -spectral sparsifier with  $O(\alpha n \log n / \varepsilon^2)$  edges in time  $O(\alpha n \log n / \varepsilon^2)$ .*

Since the algorithm **Sparsify** cannot provide the optimal sparsity when  $\alpha \gg 1$ , we will use the spectral sparsification algorithm proposed by Koutis, Levin and Peng [15] to further sparsify the graph at the end.

**Theorem 6.** [15] *There is a spectral sparsification algorithm, we call **FastSparsify** ( $G$ ), that produces a  $\varepsilon$ -spectral sparsifier with  $O(n \log n / \varepsilon^2)$  edges in time  $\tilde{O}(m \log^2 n \log(1/\varepsilon))$  with high probability.*

Using Lemma 4, Theorem 5 and Theorem 6, we can derive our main theorem:

$\overline{H} = \mathbf{SublinearSparsify}(G, \varepsilon, \delta)$
1. Let $E_{\delta n}$ be the graph given by Theorem 3.
2. Let $V_\delta$ be a random subset of $V$ with size $ E_{\delta n} $ .
3. View $E_{\delta n}$ as a graph on $V_\delta$ and let $\tilde{G}$ be the union of $G$ and $E_{\delta n}$ .
4. Let $p(u, v) = 1 / (nd_{\tilde{G}}(u)) + 1 / (nd_{\tilde{G}}(v))$ .
5. $H = \mathbf{Sparsify}(\tilde{G}, p, \Theta(n \log^2 n / \delta \varepsilon^2))$ .
6. $\overline{H} = \mathbf{FastSparsify}(H)$ .

**Theorem 7.** *Assume  $\delta \leq 1/\log n$  and  $\varepsilon < 1$  and the General Graph Model. With high probability, the **SublinearSparsify**( $G, \varepsilon, \delta$ ) algorithm produces a probabilistic  $(O(\varepsilon), O(\delta))$ -spectral sparsifier with  $O(n \log n / \varepsilon^2)$  edges in time  $\tilde{O}(n \log^4 n \log(1/\varepsilon) / \delta \varepsilon^2)$ .<sup>2</sup>*

*Proof.* Lemma 4 shows that with high probability, for all  $(u, v)$ , we have

$$p(u, v) = \frac{1}{n} \left( \frac{1}{d_{\tilde{G}}(u)} + \frac{1}{d_{\tilde{G}}(v)} \right) = \Omega \left( \frac{\delta}{\log n} \right) \frac{R_{\tilde{G}}(u, v)}{n}.$$

Hence,  $p$  satisfy the assumption of Theorem 5 with  $\alpha = O(\log n / \delta)$ . Therefore,  $H$  is a  $\varepsilon$ -spectral sparsifier of  $\tilde{G}$  with high probability. For any  $u \in \mathbb{R}^V$ , we have

$$\begin{aligned} \sum_{(x, y) \in H} \omega(x, y) (u(x) - u(y))^2 &\geq (1 - \varepsilon) \sum_{(x, y) \in \tilde{G}} (u(x) - u(y))^2 \\ &\geq (1 - \varepsilon) \sum_{(x, y) \in G} (u(x) - u(y))^2. \end{aligned}$$

---

<sup>2</sup> $\tilde{O}(f(n))$  means  $O(f(n) \log^c \log(n))$  for some constant  $c$ .

Hence,  $H$  satisfies the condition (1.2). Also, for any  $u \in \mathbb{R}^V$ , we have

$$\begin{aligned} \sum_{(x,y) \in H} \omega(x,y) (u(x) - u(y))^2 &\leq (1 + \varepsilon) \sum_{(x,y) \in \tilde{G}} (u(x) - u(y))^2 \\ &\leq (1 + \varepsilon) \sum_{(x,y) \in G} (u(x) - u(y))^2 + 4 \sum_{x \in V_\delta} (u(x))^2. \end{aligned}$$

Since  $V_\delta$  is a random subset of  $V$  with size  $\Theta(\delta n)$ , we have

$$\mathbb{E} \left( \sum_{x \in V_\delta} (u(x))^2 \right) = \Theta(\delta) \sum_{x \in V} (u(x))^2.$$

Thus, for any  $u \in \mathbb{R}^V$ , with high probability,

$$\sum_{(x,y) \in H} \omega(x,y) (u(x) - u(y))^2 \leq (1 + \varepsilon) \sum_{(x,y) \in G} (u(x) - u(y))^2 + \Theta(\delta) \|u\|^2.$$

Hence,  $H$  satisfies the condition (1.3). Therefore,  $H$  is a probabilistic  $(O(\varepsilon), O(\delta))$ -spectral sparsifier with  $O(n \log^2 n / \delta \varepsilon^2)$  edges. Using Theorem 6 and similar proof, we obtain that  $\overline{H}$  is a probabilistic  $(O(\varepsilon), O(\delta))$ -spectral sparsifier with  $O(n \log n / \varepsilon^2)$  edges.

Since the sampling probability is of the form  $1/d(s) + 1/d(t)$ , we do it by sampling each node with probability proportionally to the degree. Thus, it can be implemented in time  $O(\log n)$  using the General Graph Model.  $\square$

### 3. APPLICATIONS

In this section, we demonstrate how to apply the probabilistic spectral sparsification to solve cut-based problems. Restricting our focus on  $x \in \{0, 1\}^V$ , the upper bound (1.3) and the lower bound (1.2) of the probabilistic spectral sparsification becomes the following: Suppose  $\tilde{G}$  is a probabilistic  $(\varepsilon, \delta)$ -spectral sparsifier of  $G$ , then we have

(1) Lower Bound: We have

$$(1.1) \quad (1 - \varepsilon) \text{Cut}_G(U) \leq \text{Cut}_{\tilde{G}}(U) \quad \text{for all } U \subset V.$$

(2) Upper Bound: For all  $U \subset V$ , we have

$$(1.2) \quad \text{Cut}_{\tilde{G}}(U) \leq (1 + \varepsilon) \text{Cut}_G(U) + \delta |U| \quad \text{with high probability.}$$

The lower bound shows that any cut with a small cut value in  $\tilde{G}$  has a small cut value in  $G$  and the upper bound shows that such cut with a small cut value exists in  $\tilde{G}$  with high probability. Therefore, as long as the additive error  $\delta |U|$  is acceptable, we can approximately solve any cut-based problem on a probabilistic spectral sparsifier of the original graph and use the upper bound and lower bound to certify that it is a good solution for the original graph.

**3.1. (Uniform) Sparsest Cut Problem and Balanced Separator Problem.** The sparsest cut problem is to find a set  $U$  with  $|U| < n/2$  such that it minimizes the ratio of  $\text{Cut}_G(U)$  and  $|U|$ . The balanced separator problem is to solve the same problem with an extra condition  $|U| = \Omega(n)$ . The best known algorithm [1] for both problems achieves an  $O(\sqrt{\log n})$  approximation ratio in polynomial time. For fast algorithms, Sherman [27] gives an  $\tilde{O}(m + n^{3/2+t})$  time algorithm with approximation ratio  $O(\sqrt{\log n/t})$  and Mądry [22] gives an  $\tilde{O}(m + 2^k n^{1+1/(3 \cdot 2^k - 1) + o(1)})$  time algorithm with approximation ratio  $O(\log^{(1+o(1))(k+1/2)} n)$

for all  $k \geq 1$ . Both algorithms works for weighted graph. Using these results and our probabilistic spectral sparsifiers, we have the following:

**Corollary 8.** *Assume the graph is undirected and unweighted. For any  $t \in [O(1/\log n), \Omega(1)]$ , there is an  $\tilde{O}(n/OPT + n^{3/2+t})$  time algorithm to approximate the sparsest cut problem and the balanced separator problem with approximation ratio  $O(\sqrt{\log n/t})$ . For any integer  $k \geq 1$ , there is an  $\tilde{O}(n/OPT + 2^k n^{1+1/(3 \cdot 2^k - 1) + o(1)})$  time algorithm with approximation ratio  $O(\log^{(1+o(1))(k+1/2)} n)$ .*

*Proof.* The proof for both problems and both approximation ratios are similar. Assume it is the sparsest cut problem and we want to get an  $\alpha$  approximation algorithm. The algorithm works as follows:

- (1) Take  $\delta = 1/\log n$ .
- (2) Let  $\tilde{G}$  be a probabilistic  $(\frac{1}{2}, \delta)$ -spectral sparsifier of  $G$ .
- (3) Find an  $\alpha$  approximate sparsest cut  $\bar{U}$  on the graph  $\tilde{G}$ .
- (4) Let  $\overline{OPT}$  be the ratio of  $\text{Cut}_{\tilde{G}}(\bar{U})$  and  $|\bar{U}|$ .
- (5) If  $\delta > \overline{OPT}/2\alpha$ 
  - (a)  $\delta \leftarrow \delta/2$ , go to step 2
  - (b) Otherwise, output  $\bar{U}$ .

Let  $G$  be the original graph. Let  $U_G$  and  $OPT_G$  are an optimum set and the optimum value for this problem in graph  $G$ . Let  $OPT_{\tilde{G}}$  is the optimum value for graph  $\tilde{G}$ . Using (3.2), we have

$$OPT_{\tilde{G}} \leq \frac{\text{Cut}_{\tilde{G}}(U_G)}{|U_G|} \leq \frac{\frac{3}{2}\text{Cut}_G(U_G) + \delta|U_G|}{|U_G|} = \frac{3}{2}OPT_G + \delta.$$

Since  $\bar{U}$  is an  $\alpha$  approximate sparsest cut on  $\tilde{G}$ , we have

$$\frac{1}{\alpha}\overline{OPT} \leq OPT_{\tilde{G}} \leq \frac{3}{2}OPT_G + \delta.$$

If  $\delta < \overline{OPT}/2\alpha$ , then we have  $\overline{OPT} \leq 3\alpha OPT_G$ . Hence, (3.1) gives  $\text{Cut}_G(\bar{U})/|\bar{U}| \leq 6\alpha OPT_G$  and the set  $\bar{U}$  solve the problem in  $G$  with approximation ratio  $6\alpha$ . Otherwise, we have  $\delta$  decrease by 2. Since  $\overline{OPT} \geq \frac{1}{n}$ , the algorithm takes at most  $\log n$  iterations.  $\square$

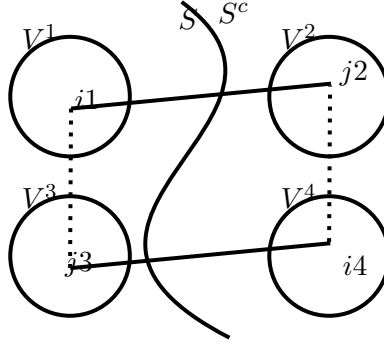
In Theorem 12, we show that the term  $n/OPT$  in running time is unavoidable. So, our reduction is almost optimal.

**3.2. Minimum s-t Cut Problem.** The Minimum s-t Cut Problem is to find a set  $U$  such that  $s \in U$ ,  $t \notin U$  and it minimizes  $\text{Cut}_G(U)$ .

**Corollary 9.** *Assume the graph is undirected and unweighted. There are an  $\tilde{O}(\sqrt{mn}/\varepsilon^3)$  time and a  $n^{1+o(1)}/\varepsilon^4$  time algorithm to find a minimum s-t cut up to an  $\varepsilon n$  error.*

*Proof.* On an undirected graph with integer weight, the proof of Theorem 4 of [17] shows an  $\tilde{O}\left(\frac{m}{\varepsilon}\sqrt{\frac{W}{n}}\right)$  time algorithm to compute an approximate minimum s-t cut with  $\varepsilon n$  additive error, where  $W$  is the total weight. Note that the total weight of the result of our sparsification is  $\tilde{O}(m)$  and changes can be made so that the weights are integers. This gives the first result.

For the second result, it follows from [13, 28].  $\square$

FIGURE 4.1. Illustration of  $G_{k,p}$ 

**3.3. Other applications.** For some cut-based problems such as the maximum cut problem and the minimum cut problem, sampling edges with constant probability gives good enough guarantee. For other cut-based problems such as the multicut problem, one can use our sparsification to reduce the problem into sparse graphs, then use the technique by Mądry [22] to further reduce the problem into almost trees, which can be then solved by elementary methods in many cases.

Our probabilistic spectral sparsifier is also useful for applications involves the graph energy  $\sum_{x \sim y} (u(x) - u(y))^2$ . It includes a lot of problems in many fields, such as approximating Fiedler vector [19], minimizing all sorts of variational problems in image processing [3].

#### 4. LOWER BOUND

In this section, we show that the additive error in upper bound (1.3) for the sparsifier is necessary. In the proof, we construct a family of random graphs and shows that it is difficult to estimate the cut value of some sets in the graphs. In the Lemma 10, we construct a family of random graphs which is served as a building block of the graphs for Theorem 11.

**Lemma 10.** *Assume the general graph model. For any integer  $k > 3$  and  $0 < p \leq 1/4$  such that  $pk^2 \geq 100$ , there is a family of random graph  $G_{k,p} = (V, E)$  with  $4k$  vertices and  $2k^2$  edges and a cut  $S \subset E$  which satisfies the following property: let  $C$  be the estimate of  $\text{Cut}(S)$  of any deterministic algorithm which calls the oracle less than  $k^2/2$  times. Then, we have*

$$\mathbb{P} \left( |C - \text{Cut}(S)| \geq k \sqrt{\frac{p}{8}} \right) \geq 0.01.$$

*Proof.* For each pair  $i, j \in [k]$ , let  $H_{ij}$  be an independent variable such that  $H_{ij} = 1$  with probability  $p$  and  $H_{ij} = 0$  otherwise. We construct a family of random graph  $G_{k,p}$  using the random variable  $\{H_{ij}\}_{i \in [k], j \in [k]}$ . The graph  $G_{k,p}$  consists of 4 sets of vertices  $V^1, V^2, V^3, V^4$  and each of them has  $k$  vertices. We call each vertex in  $V^t$  by  $it$  for some  $i \in [k]$ . If  $H_{ij} = 1$ , we place the edges  $\{(i1, j2), (j3, i4)\}$ , which is indicated by the solid lines in the figure. Otherwise, we place the edges  $\{(i1, j3), (j2, i4)\}$ . Note that this graph is  $k$  regular and hence the degree oracle does not provide any information.

Let  $S = V^1 \cup V^3$ . Then, we have  $\mathbb{E}(\text{Cut}(S)) = 2\mathbb{E}(\sum_{i,j} H_{ij}) = 2pk^2$  and  $\text{Var}(\text{Cut}(S)) = 4\text{Var}(\sum_{i,j} H_{ij}) = 4p(1-p)k^2$ . Consider any deterministic algorithm that calls the oracle less than  $k^2/2$  times. Let  $C$  be the estimate of  $\text{Cut}(S)$  given by the algorithm. Since each edge



is only affected by one random variable  $H_{ij}$ , only at most  $k^2/2$  values of  $H_{ij}$  are revealed. Let  $H$  be the set of known random variables  $H_{ij}$ . Then, we have  $|H| \leq k^2/2$ . Therefore, the cut value  $\text{Cut}(S)$  given  $H$  follows the binomial distribution  $2B(p, k^2 - |H|)$  plus the constant  $2 \sum_{ij \in H} H_{ij}$ .

Since  $p(k^2 - |H|) \geq pk^2/2 \geq 50$ , the result follows from Lemma 13.  $\square$

The following theorem shows that even the graph is quite sparse, it is not possible to improve our probabilistic spectral sparsification algorithm by too much. Instead of proving lower bound for the spectral sparsification, we show the lower bound for the cut sparsification which satisfies (1.2) and (1.3) for  $u \in \{0, 1\}^V$  only.

**Theorem 11.** *For any  $\varepsilon > 0$  and  $\delta > 0$ , it takes  $\Omega\left(\frac{n}{\varepsilon^2} + \frac{n}{\delta}\right)$  queries in the general graph model to construct a probabilistic  $(\varepsilon, \delta)$  cut sparsifier for graphs with  $n$  vertices and  $\Omega\left(\frac{n}{\varepsilon^2} + \frac{n}{\delta}\right)$  edges.*

*Proof.* We divide the proof into two cases,  $\delta < \varepsilon^2$  and  $\delta \geq \varepsilon^2$ . In both cases, we construct a family of random graphs and shows that any deterministic algorithm takes  $\Omega\left(\frac{n}{\varepsilon^2} + \frac{n}{\delta}\right)$  queries to estimate the cut value of a certain cut within the precision required.

For the first case  $\delta < \varepsilon^2$ , let  $G$  be the disjoint union of  $\delta n$  independent copies of  $G_{10\delta^{-1}, \delta^2}$  defined in Lemma 10. Let  $G_i$  be each copy and  $S_i$  be each corresponding cut defined in Lemma 10. Note that  $G$  has  $\Theta(n)$  vertices and  $\Theta(\frac{n}{\delta})$  edges.

Let us consider any deterministic algorithm which calls the oracle less than  $\frac{n}{4\delta}$  times. At least  $\frac{\delta n}{2}$  copies of  $G_i$ , the algorithm calls the oracle less than  $\frac{\delta^{-2}}{2}$  times for these  $G_i$ . Hence, Lemma 10 shows with probability 0.01, the estimate value deviates from the cut value for more than 1. For those  $S_i$ , the estimate value is either larger than the cut value by 1 or is smaller than the cut value by 1. Without loss of generality, we assume the first case happens more. And let  $\mathcal{S}$  be the set of those  $S_i$  in the first case. Then, we have  $|\mathcal{S}| = \Omega(\delta n)$  with high probability. Let  $A = \bigcup_{S \in \mathcal{S}} S$ . Then, the estimate of  $\text{Cut}(A)$  is larger than the true value by  $\Omega(\delta n)$ . Also, note that  $\text{Cut}(A) = O(\delta n)$ .

It shows that any deterministic algorithm takes at least  $\Omega(\frac{n}{\delta})$  queries to construct a probabilistic  $(O(1), \delta)$  cut sparsifier for graphs with  $n$  vertices and  $\Omega(\frac{n}{\delta})$  edges.

For the second case  $\delta \geq \varepsilon^2$ , let  $G$  be the disjoint union of  $\varepsilon^2 n$  independent copies of  $G_{10\varepsilon^{-2}, \varepsilon^2}$ . By similar argument, we can show that any deterministic algorithm takes at least  $\Omega(\frac{n}{\varepsilon^2})$  queries to construct a  $(\varepsilon, O(1))$  cut sparsifier for graphs with  $\Theta(n)$  vertices and  $\Omega(\frac{n}{\varepsilon^2})$  edges.

Combining both cases, the result follows from the Yao's principle.  $\square$

Similar lower bounds can be established for various problems. We use the sparsest cut problem as an example to show that our approach can be used to give almost optimal results.

**Theorem 12.** *For any  $O(1) > \varepsilon > \frac{1}{n}$ , it takes  $\Omega\left(\frac{n}{\varepsilon}\right)$  queries in the general graph model to distinguish between a disconnected graph and a graph with  $\min_{|U| < \frac{n}{2}} \text{Cut}(U)/|U| = \Theta(\varepsilon)$ .*

*Proof.* Let  $G^\varepsilon = G_{10n, \varepsilon n^{-1}}$  defined in Lemma 10. Put a complete graph inside  $V^1, V^2, V^3, V^4$  regions of  $G^\varepsilon$  as defined in Lemma 10. With high probability, we have  $\min_{|U| < \frac{n}{2}} \text{Cut}(U)/|U| = \Theta(\varepsilon)$ .

Since  $G^\varepsilon$  is a regular graph with same degree for all  $\varepsilon$ , the degree oracle does not provide any information. To distinguish between  $G^\varepsilon$  and  $G^0$ , the algorithm need to call the edge oracle until it found an edge from  $V^1 \cup V^3$  to  $V^2 \cup V^4$ . Since the probability of finding such edge is  $O(\varepsilon n^{-1})$ , it takes  $\Omega\left(\frac{n}{\varepsilon}\right)$  queries to distinguish between  $G^\varepsilon$  and  $G^0$ .  $\square$

## 5. ACKNOWLEDGMENTS

We thank Ronitt Rubinfeld for many helpful conversations. This work was partially supported by NSF awards 0843915 and 1111109, and Hong Kong RGC grant 2150701.

## REFERENCES

- [1] Sanjeev Arora, Satish Rao, and Umesh Vazirani. Expander flows, geometric embeddings and graph partitioning. *Journal of the ACM (JACM)*, 56(2):5, 2009.
- [2] András A Benczúr and David R Karger. Approximating st minimum cuts in  $\tilde{O}(n^2)$  time. In *Proceedings of the twenty-eighth annual ACM symposium on Theory of computing*, pages 47–55. ACM, 1996.
- [3] Tony Chan and Jianhong Shen. *Image processing and analysis: variational, PDE, wavelet, and stochastic methods*. Siam, 2005.
- [4] Benjamin Doerr. Analyzing randomized search heuristics: Tools from probability theory. *Theory of randomized search heuristics*, 1:1–20, 2011.
- [5] Alan Frieze and Ravi Kannan. Quick approximation to matrices and applications. *Combinatorica*, 19(2):175–220, 1999.
- [6] Alan M Frieze. Edge-disjoint paths in expander graphs. In *Proceedings of the eleventh annual ACM-SIAM symposium on Discrete algorithms*, pages 717–725. Society for Industrial and Applied Mathematics, 2000.
- [7] Ashish Goel, Michael Kapralov, and Sanjeev Khanna. Graph sparsification via refinement sampling. *arXiv preprint arXiv:1004.4915*, 2010.
- [8] Ashish Goel, Michael Kapralov, and Ian Post. Single pass sparsification in the streaming model with edge deletions. *arXiv preprint arXiv:1203.4900*, 2012.
- [9] David R Karger. Using random sampling to find maximum flows in uncapacitated undirected graphs. In *Proceedings of the twenty-ninth annual ACM symposium on Theory of computing*, pages 240–249. ACM, 1997.
- [10] David R Karger. Better random sampling algorithms for flows in undirected graphs. In *Proceedings of the ninth annual ACM-SIAM symposium on Discrete algorithms*, pages 490–499. Society for Industrial and Applied Mathematics, 1998.
- [11] David R Karger and Matthew S Levine. Finding maximum flows in undirected graphs seems easier than bipartite matching. In *Proceedings of the thirtieth annual ACM symposium on Theory of computing*, pages 69–78. ACM, 1998.
- [12] Jonathan Kelner, Alex Levin, et al. Spectral sparsification in the semi-streaming setting. *Leibniz International Proceedings in Informatics (LIPIcs) series*, 9:440–451, 2011.
- [13] Jonathan A Kelner, Lorenzo Orecchia, Yin Tat Lee, and Aaron Sidford. An almost-linear-time algorithm for approximate max flow in undirected graphs, and its multicommodity generalizations. *arXiv preprint arXiv:1304.2338*, 2013.
- [14] Jonathan A Kelner, Lorenzo Orecchia, Aaron Sidford, and Zeyuan Allen Zhu. A simple, combinatorial algorithm for solving sdd systems in nearly-linear time. In *Proceedings of the 45th annual ACM symposium on Symposium on theory of computing*, pages 911–920. ACM, 2013.
- [15] Ioannis Koutis, Alex Levin, and Richard Peng. Faster spectral sparsification and numerical algorithms for sdd matrices. *arXiv preprint arXiv:1209.5821*, 2012.
- [16] Ioannis Koutis, Gary L Miller, and Richard Peng. A nearly- $m \log n$  time solver for sdd linear systems. In *Foundations of Computer Science (FOCS), 2011 IEEE 52nd Annual Symposium on*, pages 590–598. IEEE, 2011.
- [17] Yin Tat Lee, Satish Rao, and Nikhil Srivastava. A new approach to computing maximum flows using electrical flows. In *STOC*, pages 755–764, 2013.
- [18] Yin Tat Lee and Aaron Sidford. Efficient accelerated coordinate descent methods and faster algorithms for solving linear systems. *arXiv preprint arXiv:1305.1922*, 2013.
- [19] Bruno Lévy. Laplace-beltrami eigenfunctions towards an algorithm that. In *Shape Modeling and Applications, 2006. SMI 2006. IEEE International Conference on*, pages 13–13. IEEE, 2006.
- [20] László Lovász. Random walks on graphs: A survey. *Combinatorics, Paul erdos is eighty*, 2(1):1–46, 1993.
- [21] Alexander Lubotzky, Ralph Phillips, and Peter Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988.
- [22] Aleksander Madry. Fast approximation algorithms for cut-based problems in undirected graphs. In *Foundations of Computer Science (FOCS), 2010 51st Annual IEEE Symposium on*, pages 245–254. IEEE, 2010.

- [23] Sharon Marko and Dana Ron. Distance approximation in bounded-degree and general sparse graphs. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*, pages 475–486. Springer, 2006.
- [24] Huy N Nguyen and Krzysztof Onak. Constant-time approximation algorithms via local improvements. In *Foundations of Computer Science, 2008. FOCS'08. IEEE 49th Annual IEEE Symposium on*, pages 327–336. IEEE, 2008.
- [25] Krzysztof Onak, Dana Ron, Michal Rosen, and Ronitt Rubinfeld. A near-optimal sublinear-time algorithm for approximating the minimum vertex cover size. In *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1123–1131. SIAM, 2012.
- [26] Michal Parnas and Dana Ron. Approximating the minimum vertex cover in sublinear time and a connection to distributed algorithms. *Theoretical Computer Science*, 381(1):183–196, 2007.
- [27] Jonah Sherman. Breaking the multicommodity flow barrier for  $o(\log n)$ -approximations to sparsest cut. In *Foundations of Computer Science, 2009. FOCS'09. 50th Annual IEEE Symposium on*, pages 363–372. IEEE, 2009.
- [28] Jonah Sherman. Nearly maximum flows in nearly linear time. *arXiv preprint arXiv:1304.2077*, 2013.
- [29] Daniel A Spielman and Nikhil Srivastava. Graph sparsification by effective resistances. *SIAM Journal on Computing*, 40(6):1913–1926, 2011.
- [30] Daniel A Spielman and Shang-Hua Teng. Spectral sparsification of graphs. *SIAM Journal on Computing*, 40(4):981–1025, 2011.
- [31] Yuichi Yoshida, Masaki Yamamoto, and Hiro Ito. An improved constant-time approximation algorithm for maximum. In *Proceedings of the 41st annual ACM symposium on Theory of computing*, pages 225–234. ACM, 2009.

## APPENDIX

**Lemma 13.** *Let  $0 \leq p \leq 1/4$  and  $n$  be an integer such that  $pn \geq 36$ . Let  $X \sim B(p, n)$ . Then, for any  $\theta$ , we have*

$$\mathbb{P}\left(|X - \theta| \geq \frac{1}{2}\sqrt{pn}\right) \geq 0.01.$$

*Proof.* Note that for any  $\theta$ , we have

$$\mathbb{P}\left(|X - \theta| \geq \frac{1}{2}\sqrt{pn}\right) \geq \mathbb{P}\left(|X - pn| \geq \frac{1}{2}\sqrt{pn}\right)$$

because of the shape of the binomial distribution. Hence, it suffices to prove the bound for  $\mathbb{P}\left(|X - pn| \geq \frac{1}{2}\sqrt{pn}\right)$ .

Using Chernoff bound, for any  $k \geq 6$ , we have

$$\begin{aligned} \mathbb{P}(|X - pn| \geq k\sqrt{pn}) &\leq 2 \exp\left(-\frac{k^2}{2 + \frac{k}{\sqrt{pn}}}\right) \\ &\leq 2 \exp(-2k). \end{aligned}$$

Hence, for  $k \geq 6$ , we have

$$\begin{aligned} \int_{|x-pn| \geq k\sqrt{pn}} (x - pn)^2 dP(x) &= 2k^2 pn \mathbb{P}(X \geq pn + k\sqrt{pn}) \\ &\quad + 4 \int_{x \geq pn + k\sqrt{pn}} (x - pn) \mathbb{P}(X \geq x) dx \\ &\leq 2k^2 pn \exp(-2k) + 4 \int_{k\sqrt{pn}}^{\infty} x \exp\left(-2\frac{x}{\sqrt{pn}}\right) dx \\ &= (2k^2 + 2k + 1)pn \exp(-2k). \end{aligned}$$

Put  $k = 6$ , we have  $\int_{|x-pn| \geq 6\sqrt{pn}} (x-pn)^2 dP \leq 0.01pn$ . Since the  $\text{Var}(X) = \int (x-pn)^2 dP = p(1-p)n \geq \frac{3}{4}pn$ , we have

$$\int_{|x-pn| < 6\sqrt{pn}} (x-pn)^2 dP \geq 0.74pn.$$

Let  $U = P(|X - pn| \geq \frac{1}{2}\sqrt{pn})$ , then we have

$$\begin{aligned} 36Un + (1-U)\frac{pn}{4} &\geq \int_{|x-pn| < 6\sqrt{pn}} (x-pn)^2 dP \\ &\geq 0.74pn. \end{aligned}$$

Hence, we have  $U \geq 0.01$ . □